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PLATES WITH BOUNDARY CONDITIONS OF ELASTIC SUPPORT

By S. J. Fuchs, M. ASCE

ENGINEERING MECHANICS DIVISION

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PAPERS

PLATES WITH BOUNDARY CONDITIONS OF ELASTIC SUPPORT

By S. J. Fuchs,[1] M. ASCE

Synopsis

A solution for the deflections, moments, and reactions in a rectangular plate under transverse load, supported at the corners, and with or without supporting beams at the panel boundaries, is presented in this paper. This solution can be classified as a unified one, since it is adaptable to most of the specific problems for which particular solutions have been developed. Provision is made for other classes of particular solutions for additional cases. The solutions for specific problems, expressed in a general form, differ only in the arbitrary constants appearing in the general form. These constants are evaluated by means of operational formulas, with dependence on parameters that are defined by the ratios of the flexural and torsional rigidities of the beams to the flexural rigidity of the plate, and by the proportions of the plate. To illustrate the procedure, the arbitrary constants are obtained for a rectangular plate under uniform load that is simply supported on beams of finite flexural rigidity on four sides. A particular value of the parameter of relative flexural rigidity for which the expressions for moment reduce to simple exact formulas also exists. Moment and reaction coefficients are evaluated at positions along the axes of symmetry and the boundaries of the simply supported plate for several values of the parameter of relative flexural rigidity. Experimental verification is obtained by use of introductory tests made on panels supported on beams of arbitrarily reduced flexural rigidity.

Introduction

Notation.—The letter symbols introduced in this paper are defined where they first appear, in the text or by illustration, and are assembled alphabetically in Appendix II for convenience of reference.

Note.—Written comments are invited for publication; the last discusson should be submitted by November 1, 1953.

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For the thin, elastic, rectangular laterally loaded plate that is supported at the corners, with or without tributary beams disposed at the panel boundaries, various important cases may be considered as corresponding to the limit values of the flexural and torsional rigidities of the beams. Thus, the clamped plate supported on rigid beams may be considered as corresponding to values of the flexural and torsional rigidities for the beams that approach infinity. Similarly, other combinations of limit values of these rigidities, which correspond to the solutions that are known for rectangular plates, are classified in Table 1, which

TABLE 1.—CLASSIFICATION DEFINED BY THE FLEXURAL AND TORSIONAL RIGIDITIES OF THE SUPPORTING BEAMS

		Flexural Rigidity, λ		
Line	Torsional Rigidity, K	$\lambda = 0$ (Plain support)	$\lambda =$ finite (Flexural support)	$\lambda \rightarrow \infty$ (Nonflexural support)
1	$K = 0$ (Simple support)	plain, simple	flexural, simple	nonflexural, simple
2	$K =$ finite (Torsional support)	plain, torsional	flexural, torsional	nonflexural, torsional
3	$K \rightarrow \infty$ (Clamped support)	plain, clamped	flexural, clamped	nonflexural, clamped

applies to polygons in general. Therefore, a solution for the laterally loaded plate, involving formulas expressing a dependence on parameters corresponding to the torsional and flexural rigidities of the beams, may be considered a general solution, since it would reduce to the known solutions for limit values of the parameters. The solution would also yield classes of solutions for intervening ranges of parametric values. The derivation and application of such a solution, with particular reference to the rectangular plate, will be investigated in this paper.

Historical Note.—The extensive and distinguished literature on the theory of plates can be acknowledged only inadequately by the brief list of references that are limited to a few distinct solutions, exclusive of difference equation methods, which have been applied to a majority of the historic problems of thin, rectangular, transversely loaded plates.

The first of these solutions, for the deflection of panels on simple, nonflexural supports, was introduced by C. L. M. H. Navier[2] in 1820, and comprised a double infinite series of trigonometric terms. Other investigators have applied the same form of solution to the clamped plate, with or without nonflexural supports.

The next solution to appear, which was also important in the sense of initiating a new class of applications, was that of M. Lévy,[3] in 1899. The latter's primitive for the applicable differential equation may be expressed as $w = \sum_m Y_m \cos mx + \phi$, in which w is the deflection of the plate; m is an index of

[2] "Extrait des Recherches sur la Flexion des Plans élastiques," by C. L. M. H. Navier, *Bulletin des Sciences*, Société Philomathique de Paris, Vol. 10, 1823, pp. 94–102.

[3] "Sur l'Équ ibre élastique d'une Plaque rectangulaire," by M. Lévy, *Comptes Rendus*, Vol. 129, 1889, pp. 535–539.

the summation; Y_m, in the complementary function, comprises hyperbolic functions in the variable y; x is the coordinate variable; and ϕ is a particular integral. Two applications of Lévy's result, by H. Hencky[4] and A. Nadai,[5] are also examples of a construction which often recurs in the literature. These applications involve the superposition of a pair of expressions, each of the same form, but with x and y, and a and b, mutually interchanged, in which a and b are the panel span lengths, measured parallel to the x and y axes, respectively. Mr. Hencky employed a simple superposition of functions $f(x, y, a, b)$ and $f(y, x, b, a)$ in which f comprised Lévy's primitive, for the problem of the clamped, nonflexurally supported panel. Mr. Nadai's method consisted in the construction of an orthonormalized set of functions w_k out of linear combinations of functions $w_n = u_n(x, y, a, b) + v_n(y, x, b, a)$, in which u_n and v_n essentially comprise the form of Lévy's solution.

By a different mathematical method (that of the variational calculus) a third method was introduced by W. Ritz.[6] In this method, a series of functions was set up which satisfied the boundary conditions, and the coefficients of these functions were derived by minimum energy considerations. When, however, as in this investigation, the difficulty lies in the formulation of expressions that satisfy the boundary conditions, this method does not appear advantageous.

To satisfy the range of boundary conditions considered in this investigation, it is sufficient to use any convenient primitive of the differential equation which contains arbitrary constant coefficients. Of the known solutions, the primitive of Lévy is available for this purpose and may be conveniently utilized.

Formulation of the Solution

In the formulation of the solution the Kirchoff-Kelvin-Tait theory of boundary conditions is presupposed.[7] It has been shown[8] that, compared with more exact assumptions, the possible error by this theory is small for thin plates and the error is confined to the narrow layers adjacent to the boundaries. By this theory two equations for each boundary edge are required in order to define the boundary conditions for a polygonal plate, or $2n$ equations for a polygon of n sides. Furthermore, the plate is assumed to be thin, to have small deflections, and the middle surface is assumed to be inextensional, so that sides of elements that are normal before bending remain normal after bending. In agreement with these assumptions, the differential equation used is the fourth order biharmonic or plate equation

$$N\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = p \qquad (1)$$

in which N is the flexural rigidity of the plate and p is the intensity of the dis-

[4] "Der Spannungszustand in rechteckigen Platten," by H. Hencky, Oldenbourg, Munich, Germany, 1913.

[5] "Die Elastischen Platten," by A. Nadai, Springer, Berlin, Germany, 1925, pp. 180–184.

[6] "Über eine neue Methode zur Losung gewisser Variationsprobleme der mathematischen Physik," by W. Ritz, *Crelles Journal*, Vol. 135, 1909, pp. 1–61.

[7] "Treatise on Natural Philosophy," by Lord Kelvin (W. Thomson) and P. G. Tait, University Press, Cambridge, England, Part 11, 1883, pp. 188–192.

[8] "The Edge Effect in the Bending of Plates," by K. O. Friedrichs, in "Reissner Anniversary Volume: Contributions to Applied Mechanics," J. W. Edwards, Ann Arbor, Mich., 1949.

tributed load. Since the complementary function or solution of the homogeneous portion of this equation includes four constants of integration, $\frac{2n}{4}$ independent solutions in superposition are required to yield sufficient constants for application to the $2n$ equations, namely $w_1 + w_2 + \cdots + w_{n/2}$. In the case of the parallellogram and, in particular, that of the rectangle $w = w_1 + w_2 + \phi$, in which w_1 and w_2 may be of the same form but with x, y, and a, b mutually interchanged; that is, $w_1 = f(x, y, a, b)$ and $w_2 = f(y, x, b, a)$.

Therefore both w_1 and w_2 are solutions of Eq. 1, but w_2 may be considered as resulting from a rotation of Cartesian axes from those of w_1. Since Eq. 1 is an invariant differential equation under rotation of axes, w_2 is of the same form as w_1, when referred to the new axes. More explicitly, denoting the new coordinates by primes, and with rotation through $-\pi/2$—

$$w_2 = f(x', y', a', b') = f(-y, x, -b, a) = f(y, x, b, a) \qquad (2)$$

—the negative signs being included in the arbitrary constants implicitly contained in the last equation.

For convenience, expressions similar to w_1 and w_2 which occur in the same form, but with x, y or a, b, or both, mutually interchanged, may be termed as being analogous to each other.

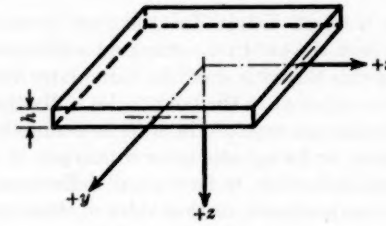

Fig. 1.—Orientation of the Plate and the Coordinate Axes

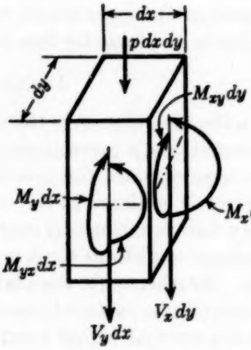

Fig. 2.—The Positive Directions of the Stress Resultants

The process of superposition of component solutions referred to successively rotated Cartesian axes, may be extended to the construction of a formal solution for the general polygon. However, the analysis contained in this paper is confined to symmetrical conditions of the rectangular plate. The boundaries and orientation of the plate are shown in Fig. 1 and the positive sense of the stress resultants is shown in Fig. 2. The flexural rigidity of the plate may be defined as $N = \frac{E\,h^3}{12}(1 - \mu^2)$, in which E represents the modulus of elasticity of the plate material; h is the thickness of the plate; and μ denotes Poisson's ratio for the plate material.

A complementary function of Eq. 1 utilized as a component solution to Eq. 2 is the primitive of Lévy:

$$w_1 = \frac{a^4}{N}\sum_m^\infty (A_{2m} \cosh \alpha\, y + B_{2m}\, \alpha\, y \sinh \alpha\, y) \cos \alpha\, x \qquad (3)$$

in which $\alpha = m\,\pi/a$, the index m being assumed to include all integers. Accordingly, application of Eq. 3 and its analogous expression in the expression for w yields, for symmetrical conditions of the rectangular plate under any symmetrical loading,

$$w = \frac{a^4}{N}\sum_m^\infty (A_{2m} \cosh \alpha\, y + B_{2m}\, \alpha\, y \sinh \alpha\, y) \cos \alpha\, x + \frac{b^4}{N}\sum_m^\infty (A_{1m} \cosh \beta\, x + B_{1m}\, \beta\, x \sinh \beta\, x) \cos \beta\, y + k_1\, x^2 + k_2\, y^2 + k_0 + \phi \qquad (4)$$

in which $\beta = m\,\pi/b$; the terms containing the arbitrary constants k_1, k_2, and k_0 are unaffected by the differential operator, and are added for greater generality; and A_{1m}, A_{2m}, B_{1m}, and B_{2m} are arbitrary constant coefficients corresponding to each value of the index m of the series comprising the primitive of the plate differential equation.

Eq. 4, for each value of the index m, contains four arbitrary constants for application to the equations which define boundary conditions. As will be shown subsequently, all the classifications of Table 1 may be expressed by four equations, for either the symmetrical or antisymmetrical conditions of the rectangular plate. Accordingly, Eq. 4 is the solution embracing all the classifications of the boundary conditions listed in Table 1, subject to the determination of the corresponding four sets of constants, A_{2m}, A_{1m}, B_{2m}, and B_{1m}, consistent with the boundary conditions assigned.

It is evident that equations similar to Eq. 4 can be set up for the antisymmetrical and also for the general or nonsymmetrical plate conditions.

The set of four equations, which represents all the classifications of Table 1, expresses the equilibrium of the plate at its boundary edges. These equations may be written as[9]

$$[w_{yyy} + (2 - \mu)\, w_{xxy}]_{y=\pm b/2} = \lambda_2\, (w_{xxxx})_{y=\pm b/2} \qquad (5)$$

$$[w_{xxx} + (2 - \mu)\, w_{yyx}]_{x=\pm a/2} = \lambda_1\, (w_{yyyy})_{x=\pm a/2} \qquad (6)$$

$$[w_{yy} + \mu\, w_{xx}]_{y=\pm b/2} = \kappa_2\, (w_{xxy})_{y=\pm b/2} \qquad (7)$$

$$[w_{xx} + \mu\, w_{yy}]_{x=\pm a/2} = \kappa_1\, (w_{yyx})_{x=\pm a/2} \qquad (8)$$

Eqs. 5 and 6 are statements for the equilibrium of the transverse reaction between the plate and the beams at the edges $y = \pm b/2$ and $x = \pm a/2$, respectively. The parameters λ_2 and λ_1 denote the ratios of the flexural rigidities of the beams to the plate at the edges $y = \pm b/2$ and $x = \pm a/2$; that is $\lambda_2 = E_2\, I_2/N$, $\lambda_1 = E_1\, I_1/N$, in which E_2 and E_1 denote the moduli of elas-

[9] "Theory of Plates and Shells," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York., N. Y., 1940, p. 89.

ticity, and I_2 and I_1 the moments of inertia, of the corresponding beams. It is assumed that only forces transverse to the plane of the plate are transmitted between the plate and beams.

Eqs. 7 and 8 are expressions for the equilibrium in the moment between the plate and the beams about the axes coinciding with the edges $y = \pm b/2$, $x = \pm a/2$, respectively. The parameters κ_2 and κ_1 denote the ratios of the torsional rigidities of the beams to the flexural rigidity of the plate at the edges $y = \pm b/2$, $x = \pm a/2$; that is, $\kappa_2 = H_2/N$ and $\kappa_1 = H_1/N$. It is assumed that longitudinal stresses caused by warping restraint may be neglected in the beams under torsion.

The various classifications in Table 1 correspond to setting at zero or allowing the values of λ_2 and λ_1, or κ_2 and κ_1, or both pairs of parameters to approach infinity. The central position in the table corresponds to all the finite values for all of these parameters.

Eqs. 5 to 8 are the means for the determination of the constant coefficients for the solution of the problem. The process could be used in a straightforward manner as a means of obtaining a solution having general application to all the classifications of Table 1. However, to simplify the presentation, the method is applied to the simply, flexurally supported panel, corresponding to line 1 of Table 1.

Simply, Flexurally Supported Rectangular Plates Under Uniform Load

The formal solution for the simply, flexurally supported rectangular plates under uniform load is the inclusive solution given by Eq. 4. The particular integral utilized, corresponding to the uniform loading assigned, is

$$\phi = \frac{2\,p}{a\,N} \sum_{m=1,3,\cdots}^{\infty} s(m)\,\alpha^{-5} \cos \alpha\, x + \frac{2\,p}{b\,N} \sum_{m=1,3,\cdots}^{\infty} s\,(m)\,\beta^{-5} \cos \beta\, y \ldots (9)$$

in which $s(m)$ is a convenient method of representing the alternating sign; that is $s(m) = (-1)^{(m-1)/2}$, with m equal to $1,3,\cdots$ in succession. The problem reduces to one of determining the values of the arbitrary constants contained in the solution, consistent with Eqs. 5 to 8, by setting $\kappa_1 = \kappa_2 = 0$. Accordingly, Eq. 7 becomes for simple suport,

$$(w_{yy} + \mu\, w_{xx})_{y=\pm b/2} = 0 \ldots\ldots\ldots\ldots (10)$$

Substitution in Eq. 10, which is an equation for the derivatives of Eq. 4, yields

$$- s(m)\,\frac{2\,\mu\,p}{a\,N} \sum_{m=1,3,\cdots}^{\infty} \alpha^{-3} \cos \alpha\, x + 2\,k_2 + 2\,k_1\,\mu + \frac{a^4}{N} \sum_{m=0,1,2,\cdots}^{\infty} \times \alpha^2 \left\{ \left[(1-\mu)\,A_{2m} + 2\,B_{2m} \right] \cosh \frac{b}{2}\,\alpha + (1-\mu)\,B_{2m}\,\frac{b}{2}\,\alpha \sinh \frac{b}{2}\,\alpha \right\} \cos \alpha\, x = 0 \ldots (11)$$

Eq. 11 can be regarded as a sum of an infinite set of equations, one equation for each value of m, and each equation containing, in all its terms, a factor comprising a trigonometric function of similar index. On elimination of this common factor, each equation is solved for the corresponding constant, A_{2m}, which results in

$$A_{2m} = \frac{s(m)\ 2\ \mu\ p \operatorname{sech} \frac{b}{2} \alpha}{a^5\ \alpha^5\ (1 - \mu)} - \left[\frac{2}{1 - \mu} + \frac{b}{2} \alpha \tanh \frac{b}{2} \alpha\right] B_{2m} \quad \text{(12a)}$$

The evaluation of A_{2m} expressed by Eq. 12a and the trivial values $k_2 = k_1 = 0$ are sufficient to satisfy Eq. 11 for all values of x. Thus the monomial terms containing k_2 and k_1 are eliminated from the solution for problems of simple support.

Similarly, the equation analogous to Eq. 12a, obtained by application of Eq. 8 for the edge at $x = \pm\, a/2$, is

$$A_{1m} = \frac{s(m)\ 2\ \mu\ p \operatorname{sech} \frac{a}{2} \beta}{b^5 \beta^5\ (1 - \mu)} - \left(\frac{2}{1 - \mu} + \frac{a}{2} \beta \tanh \frac{a}{2} \beta\right) B_{1m} \quad \text{(12b)}$$

Eq. 5 becomes, on substitution of the appropriate derivatives of w,

$$\sum_{m=1,3,\cdots}^{\infty} \frac{2\,p}{b\,\beta^2} + \sum_{m=0,1,2,\cdots}^{\infty} a^4\,\alpha^3 \left\{\left[(\mu - 1)\ A_{2m} + (1 + \mu)\ \beta_{2m}\right] \sinh \frac{b}{2} \alpha\right.$$

$$\left. + (\mu - 1)\ B_{2m} \frac{b}{2} \alpha \cosh \frac{b}{2} \alpha\right\} \cos \alpha\, x - \sum_{m=1,3,\cdots}^{\infty} \frac{s(m)\ 2\ p}{a\ \alpha} \lambda_2 \cos \alpha\, x$$

$$- \sum_{m=0,1,2,\cdots}^{\infty} \lambda_2\, a^4\, \alpha^4 \left[A_{2m} \cos \frac{b}{2} \alpha + B_{2m} \frac{b}{2} \alpha \sinh \frac{b}{2} \alpha\right] \cos \alpha\, x$$

$$+ \sum_{m=0,1,2,\cdots}^{\infty} s(m)\, b^4\, \beta^3 \left\{\left[(\mu - 1)\ A_{1m} - 2\ (2 - \mu)\ B_{1m}\right] \cosh \beta\, x\right.$$

$$\left. + (\mu - 1)\ B_{1m}\, \beta\, x \sinh \beta\, x\right\} = 0 \quad \text{(13)}$$

The series of terms of Eq. 13, comprising mixed hyperbolic, trigonometric, and other functions, do not enable the solving for the unknown constants in a direct manner. However, by a process utilized by Hencky[4] among others, the mixed terms are first expressed in equivalent trigonometric series, with a period of $2a$, which results in

$$\sum \cosh \beta\, x = \frac{2}{a} \sum \sum a_{1mn} \cos \alpha'\, x \quad \text{(14a)}$$

$$\sum x \sinh \beta\, x = \frac{2}{a} \sum \sum b_{1mn} \cos \alpha'\, x \quad \text{(14b)}$$

and

$$\frac{2\,p}{b} \sum_{m=1,3,\cdots}^{\infty} \frac{1}{\beta^2} = \frac{8\,p}{a\,b} \sum_{m=1,3,\cdots}^{\infty} \sum_{n=1,3,\cdots}^{\infty} \frac{s(n)}{\beta^2\,\alpha'} \cos \alpha'\, x \quad \text{(14c)}$$

in which $s(n) = (-1)^{(n-1)/2}$, $\alpha' = n\,\pi/a$, and the indices are assumed to comprise all integers, except as otherwise indicated.

The order of summation of the double series shown in Eqs. 14 may be reversed by interchanging the indices m and n. On substitution in Eq. 13 of these series, with interchanged indices, and on elimination of the factor $\cos \alpha x$, the equation for each odd value of m is

$$\left[a^4 \alpha^3 (\mu - 1) \sinh \frac{b}{2}\alpha - \lambda_2 a^4 \alpha^4 \cosh \frac{b}{2}\alpha\right] A_{2m} + \left\{a^4 \alpha^3 \left[(1 + \mu) \right.\right.$$

$$\left.\left. \times \sinh \frac{b}{2}\alpha + (\mu - 1) \frac{b}{2}\alpha \cosh \frac{b}{2}\alpha\right] - \lambda_2 \frac{b}{2} a^4 \alpha^5 \sinh \frac{b}{2}\alpha\right\} B_{2m}$$

$$+ 2 \sum_{n=0,1,2,\cdots}^{\infty} \operatorname{sgn}' a^2 (\alpha')^3 \{b\,[(\mu - 1) A_{1n} - 2 (2 - \mu) B_{1n}]\, a_{1nm}$$

$$+ n (\mu - 1) B_{1n} b_{1nm}\} = s(m) \frac{2 \lambda_2 p}{a \alpha} - \frac{s(m) 8 p}{a^2 \beta} \sum_{n=1,3,\cdots}^{\infty} \frac{1}{(\alpha')^2}$$

$$= s(m) \frac{2 \lambda_2 p}{a \alpha} - s(m) \frac{p}{\beta} \quad \text{..(15)}$$

For each even value of the index m, the corresponding equation is homogeneous, that is the right-hand side of Eq. 15 vanishes. Hence, it is sufficient to satisfy the equations for the boundary conditions by the trivial solutions for the constants of even index; that is,

$$A_{1m} = B_{1m} = A_{2m} = B_{2m} = 0, \quad m = 0, 2, 4, \cdots \quad \text{...........(16)}$$

Accordingly, the Fourier coefficients a_{1nm} and b_{1nm} of Eqs. 14, with interchanged indices, are evaluated only for odd values of m and n, corresponding to Eq. 15, by use of the expressions

$$a_{1nm} = \frac{s(m)\, 2\, \alpha \cosh \frac{a}{2}\beta'}{(\alpha')^2 \left(\frac{m^2}{n^2} + \frac{a^2}{b^2}\right)} \quad \text{..................(17a)}$$

$$b_{1nm} = \frac{s(m)\, b\, \beta \sinh \frac{a}{2}\beta'}{(\alpha')^2 \left(\frac{m^2}{n^2} + \frac{a^2}{b^2}\right)} - \frac{s(m)\, 4\, \beta \cosh \frac{a}{2}\beta'}{(\alpha')^3 \left(\frac{m^2}{n^2} + \frac{a^2}{b^2}\right)^2} \quad \text{..........(17b)}$$

These expressions for the Fourier coefficients and Eqs. 12 for A_{1m} and A_{2m} are simultaneously introduced into Eq. 15, and the terms resulting as coefficients of B_{2m} are simplified. On further simplification of the result by use of the summation

$$\sum_{n=1,3,\cdots}^{\infty} \frac{1}{n^4 \pi^4 \left(\frac{a^2}{b^2} + \frac{m^2}{n^2}\right)} = - \frac{a}{4\, m^3 \pi^3 b}\left(\tanh \frac{m \pi b}{2 a} - \frac{m \pi b}{2 a}\right) \quad \text{...(18)}$$

and by the notation

$$B_{2m} = s(m)\, m^2 \pi^2 \cosh \frac{m \pi b}{2a} \quad \text{(19a)}$$

$$B_{1m} = s(m)\, m^2 \pi^2 \cosh \frac{m \pi a}{2b} \quad \text{(19b)}$$

$$\Gamma_{2m} = m \pi (3 + \mu) \tanh \frac{m \pi b}{2a} + (\mu - 1) \frac{m^2 \pi^2 b}{2a} \operatorname{sech}^2 \frac{m \pi b}{2a} + \frac{2 m^2 \pi^2 \lambda_2}{(1 - \mu) a} \quad \text{(19c)}$$

and

$$\Gamma_{1m} = m \pi (3 + \mu) \tanh \frac{m \pi a}{2b} + (\mu - 1) \frac{m^2 \pi^2 a}{2b} \operatorname{sech}^2 \frac{m \pi a}{2b} + \frac{2 m^2 \pi^2 \lambda_1}{(1 - \mu) b} \quad \text{(19d)}$$

the dimensionless form derived from Eq. 15 is

$$B_{2m} - \frac{8(1 - \mu) b}{\Gamma_{2m} a} \sum_{n=1,3,\cdots}^{\infty} \frac{m^3}{n^3 \left(\frac{a^2}{b^2} + \frac{m^2}{n^2} \right)^2} B_{1n} = \frac{2 \lambda_2 2p}{m \pi (1 - \mu) a \Gamma_{2m}} - \frac{(1 - \mu) b p}{m \pi a \Gamma_{2m}}, \quad m = 1, 3, \cdots \quad \text{(20a)}$$

An equation analogous to Eq. 20*a* can be derived from Eq. 6, and is of the form

$$B_{1m} - \frac{8(1 - \mu) a}{\Gamma_{1m} b} \sum_{n=1,3,\cdots}^{\infty} \frac{m^3}{n^3 \left(\frac{b^2}{a^2} + \frac{m^2}{n^2} \right)^2} B_{2n} = \frac{2 \lambda_1 2p}{m \pi (1 - \mu) b \Gamma_{1m}} - \frac{(1 - \mu) p}{m \pi b \Gamma_{1m}}, \quad m = 1, 3, \cdots \quad \text{(20b)}$$

The constants B_{1m}, B_{2m}, with m equal to $1, 3, \cdots$, which, with the related constants A_{1m} and A_{2m} were required to satisfy the equations for the boundary conditions assigned, comprise the formal solution of the infinite sets of numerical equations represented by Eqs. 20.

In practice, the unknowns of Eqs. 20 are approximated by solving, in a conventional manner, partial sets of the numerical equations. An illustration of the procedure for the numerical application of Eqs. 20 is provided by the following example, that corresponds to an arbitrarily assigned value of the parameter λ.

Numerical Example.—To simplify the example, the plate is assumed to be square, and the sides are equal to a. The parametric value assigned to the plate is $\frac{\lambda}{a} = (1 - \mu) \frac{1}{4}$, in which $\mu = 0.15$. For the square plate, Eqs. 20

reduced to the same form, which for the numerical values assumed become

$$B_m - \frac{6.8\, m^3}{\Gamma_m} \sum_{n=1,3,\cdots}^{\infty} \frac{1}{n^3\left(1 + \dfrac{m^2}{n^2}\right)^2} B_n = -\frac{35}{m\,\pi\,\Gamma_m}, \quad m = 1, 3, \cdots \ldots \text{(21}a\text{)}$$

A partial set of numerical equations is obtained from Eq. 21a, corresponding to the first six values of the index m; that is $m = 1, 3, \cdots 11$. For a load of $p = 1$, these equations become

$$\begin{bmatrix} 0.8726 & -0.0153 & -0.0038 & -0.0014 & -0.0007 & -0.0004 \\ -0.0248 & 0.9771 & -0.0107 & -0.0052 & -0.0028 & -0.0016 \\ -0.0073 & -0.0128 & 0.9902 & -0.0063 & -0.0039 & -0.0025 \\ -0.0030 & -0.0067 & -0.0068 & 0.9945 & -0.0040 & -0.0029 \\ -0.0015 & -0.0038 & -0.0045 & -0.0042 & 0.9965 & -0.0027 \\ -0.0009 & -0.0023 & -0.0030 & -0.0031 & -0.0028 & 0.9976 \end{bmatrix} \begin{bmatrix} B_1 \\ B_3 \\ B_5 \\ B_7 \\ B_9 \\ B_{11} \end{bmatrix} = \begin{bmatrix} -0.8350\times10^{-2} \\ -0.5015\times10^{-3} \\ -0.1289\times10^{-3} \\ -0.5115\times10^{-4} \\ -0.2533\times10^{-4} \\ -0.1435\times10^{-4} \end{bmatrix} \ldots \text{(21}b\text{)}$$

This set of numerical equations may be represented algebraically by the relationship $l\,c = r$, in which l is the matrix of the numerical coefficients on the left-hand side of Eqs. 21b, c is the column matrix comprising the unknowns B_m, with $m = 1, 3, \cdots$, and r is the column on the right-hand side of Eqs. 21b. The solution for c may be obtained by an iterative operational formula first reported by J. Morris,[10] and of the form:

$$c = l^{-1}\,r = (I + t + t^2 + \cdots)\,r \ldots\ldots\ldots\ldots\ldots \text{(22}a\text{)}$$

provided the matrix series indicated converges, in which t is the matrix l minus the unit diagonal; that is $t = l - I$. Accordingly, substitution of the corresponding matrices into Eq. 22a yields successively

$$\begin{array}{ccccccccc} r & + & tr & + & t(tr) & + & t(t^2r) & + & t(t^3r) \end{array} = c = B_m$$

$$\begin{bmatrix} -0.8350 & -0.1072 & -0.0140 & -0.0019 & -0.0002 \\ -0.5015 & -0.2202 & -0.0326 & -0.0043 & -0.0006 \\ -0.1289 & -0.0691 & -0.0116 & -0.0016 & -0.0002 \\ -0.5115 & -0.2971 & -0.0541 & -0.0077 & -0.0010 \\ -0.2533 & -0.1535 & -0.0296 & -0.0043 & -0.0006 \\ -0.1435 & -0.0931 & -0.0184 & -0.0027 & -0.0004 \end{bmatrix} \begin{bmatrix} -0.9581\times10^{-2} \\ -0.7591\times10^{-3} \\ -0.2114\times10^{-3} \\ -0.8715\times10^{-4} \\ -0.4412\times10^{-4} \\ -0.2580\times10^{-4} \end{bmatrix} \ldots \text{(22}b\text{)}$$

These values of the unknowns, on substitution into Eqs. 21b agree to within one unit in the last significant digit of the respective right-hand side. The corresponding constants B_m and A_m, with $m = 1, 3, \cdots 11$, are computed by use of Eqs. 19 and 12 and converge rapidly, as follows

$$(B_m) = (-0.3869\times10^{-3},\ 0.1535\times10^{-6},\ -0.6652\times10^{-9},\ 0.6047\times10^{-11},\ -0.8001\times10^{-13},\ 0.1354\times10^{-14})$$

$$(A_m) = (0.1927\times10^{-2},\ -0.1170\times10^{-5},\ 0.7076\times10^{-8},\ -0.8302\times10^{-10},\ 0.1348\times10^{-11},\ -0.2702\times10^{-13})$$

These numerical values, multiplied by the applied load p, are inserted for the corresponding constants into Eq. 4. The result is the evaluation of w as a partial sum for the assigned values of the parameter λ.

[10] "On a Simple Method for Solving Simultaneous Linear Equations by a Successive Approximation Process," by J. Morris, *Journal*, Royal Aeronautical Soc., Vol. 39, 1935, p. 349.

Corroboration by Known Solutions.—It can be verified that for $\lambda_1 \to \infty$ and $\lambda_2 \to \infty$, Eqs. 20 reduce to expressions that are known in literature for this limit case of nonflexural support.

Thus for $\lambda_2 \to \infty$, from Eq. 20*a*,

$$B_{2m} = \frac{p}{m^3 \pi^3} \qquad (23a)$$

$$B_{2m} = s(m) \frac{p}{m^5 \pi^5} \operatorname{sech} \frac{m \pi b}{2 a} \qquad (23b)$$

and from Eq. 12*a*

$$A_{2m} = \frac{-s(m)^p \left(2 + \frac{m \pi b}{2 a} \tanh \frac{m \pi b}{2 a}\right)}{m^5 \pi^5 \cosh \frac{m \pi b}{2 a}} \qquad (23c)$$

Similarly, expressions analogous to Eqs. 23*b* and 23*c* for the constants B_{1m} and A_{1m} are derived by use of Eqs. 20*b* and 12*b*. These expressions for the values of B_{2m}, B_{1m}, A_{2m}, and A_{1m}, together with the formal expression for w in Eq. 4, complete the required solution. However, for the nonflexural case, Eq. 4 may be simplified since each solution for w_1 and w_2 separately satisfies the boundary conditions on all four sides. Since there can be only a unique solution, $w_1 = w_2 = w/2$, which is

$$2 p a^{-1} \sum_{m=1,3,\cdots}^{\infty} s(m)\, \alpha^{-5} \cos \alpha x + a^4 \sum_{m=1,3,\cdots}^{\infty} (A_{2m} \cosh \alpha y + B_{2m}\, \alpha y \sinh \alpha y) \cos \alpha x = 2 p b^{-1} \sum_{m=1,3,\cdots}^{\infty} s(m)\, \beta^{-5} \cos \beta y + b^4 \sum_{m=1,3,\cdots}^{\infty} (A_{1m} \cosh \beta x + B_{1m}\, \beta x \sinh \beta x \;\; \cos \beta y \qquad (24)$$

in which the constants A_{2m} and B_{2m} are evaluated by Eqs. 23*b* and 23*c* and A_{1m} and B_{1m} by the analogous expressions. Thus, the left-hand side of Eq. 24, multiplied by two, agrees with a classical solution for this case of nonflexural support.[11]

It can be shown that for the classification $\lambda_2 \to \infty$, and λ_1 finite, Eqs. 20 also reduce to corresponding expressions derived in the literature.

Other Assigned Loading.—For other load functions, particular integrals expressed as infinite trigonometric series may be introduced for ϕ in the formal solution, and equations similar to Eqs. 20 can be obtained by the same method of derivation. The left-hand sides of Eqs. 20 remain the same for all particular integrals expressed as infinite trigonometric series, evaluated for odd integer values of the index.

[11] "Theory of Plates and Shells," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1940, p. 125.

Numerical Evaluations of Moments and Reactions

Numerical evaluations were made by the author of various moment and reaction coefficients for square plates over the full range of relative rigidities of the supporting beams. Rectangular plates, simply supported and uniformly loaded, were also investigated.

For square plates, values of the parameter λ were selected at intervals in the range from $\lambda = 0$ to $\lambda \rightarrow \infty$, and corresponding sets of computations of the constants B_m and A_m were carried through as described previously. However, for greater accuracy, summations were carried for ten values of the index, and sets of ten numerical equations in each case were solved for the constants $\mathbf{B}_m$ with $m = 1, 3, \cdots, 19$, by method of successive elimination as adapted for computing machines by P. D. Crout.[12]

Sets of moment and reaction coefficients were computed for ten distinct stations along the axes of symmetry, along the diagonals, and along the edges of the panel. The differential expressions used for the computation of the moments M_y comprise the left-hand side of Eq. 7. Similarly, M_x is expressed by the left-hand side of Eq. 8. The differential expressions used for the reaction at the edges $y = \pm b/2$ and $x = \pm a/2$ comprise, respectively, the left-hand sides of Eqs. 5 and 6. The evaluation for each station of the derivatives of w, appearing in the differential expressions, was effected by substitution of the constants A_m and B_m in the respective expressions for the derivatives of w, and by summation for values of $m = 1, 3, \cdots, 19$.

The results of these computations are shown in Table 2 and Figs. 3 to 7, in which $\mu = 0.15$.

TABLE 2.—Moment Coefficients and Deflections[a]

	Relative Stiffness λ/a						
	0.0	0.10	0.25	0.36125	1.0	3.0	∞
C_{mxy}[b]	0.125	0.0635	0.0317	0.0187	−0.01082	−0.02834	−0.03941
$\frac{w_m}{p\,a^4/N}$	0.026340	0.020587	0.015981	0.013021	0.009044	0.006015	0.004062

[a] For square, simply supported plates under uniform load p. [b] $C_{mxy} = \frac{M_{xy}}{p\,a^2}$. Moment coefficients refer to twisting moments of the corners of the panel and the deflections are at the center of the panel.

A Particular Solution.—For panels corresponding to the parametric values

$$\frac{\lambda_2}{a} = \frac{(1-\mu)^2\, b}{2\,a} \qquad (25a)$$

and

$$\frac{\lambda_1}{b} = \frac{(1-\mu)^2\, a}{2\,b} \qquad (25b)$$

it is found that the various expressions for moment and reaction reduce to simple exact formulas. Thus, in all cases which correspond to Eqs. 25, the right-hand

[12] "A Short Method for Evaluating Determinants and Solving Systems of Linear Equations with Real or Complex Coefficients," by P. D. Crout, *Transactions*, Am. Inst. of Electrical Engrs., Vol. 60, 1941, pp. 1235–1240.

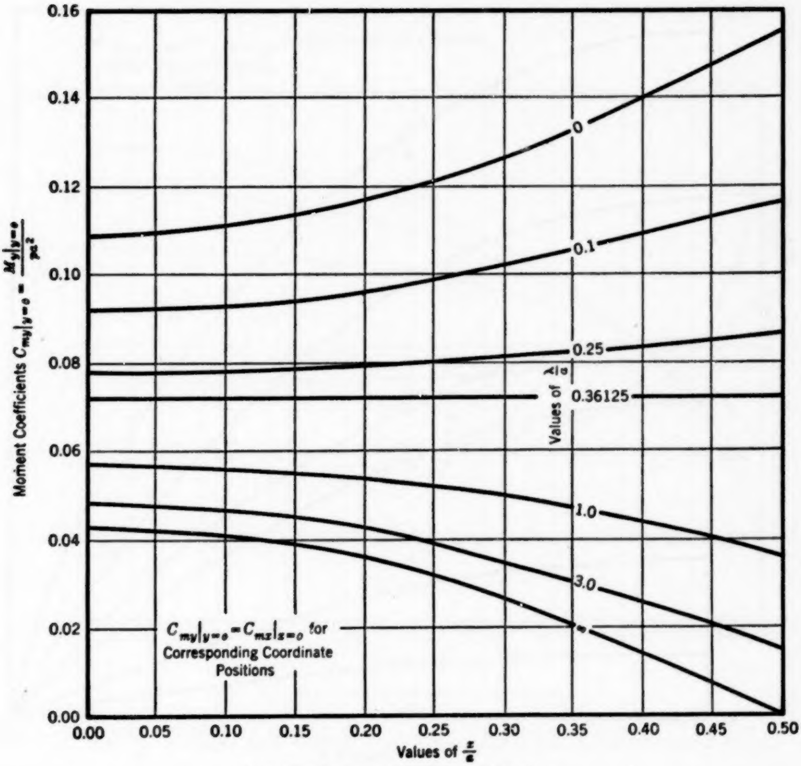

Fig. 3.—Moment Coefficients $C_{my}|_{y=0}$

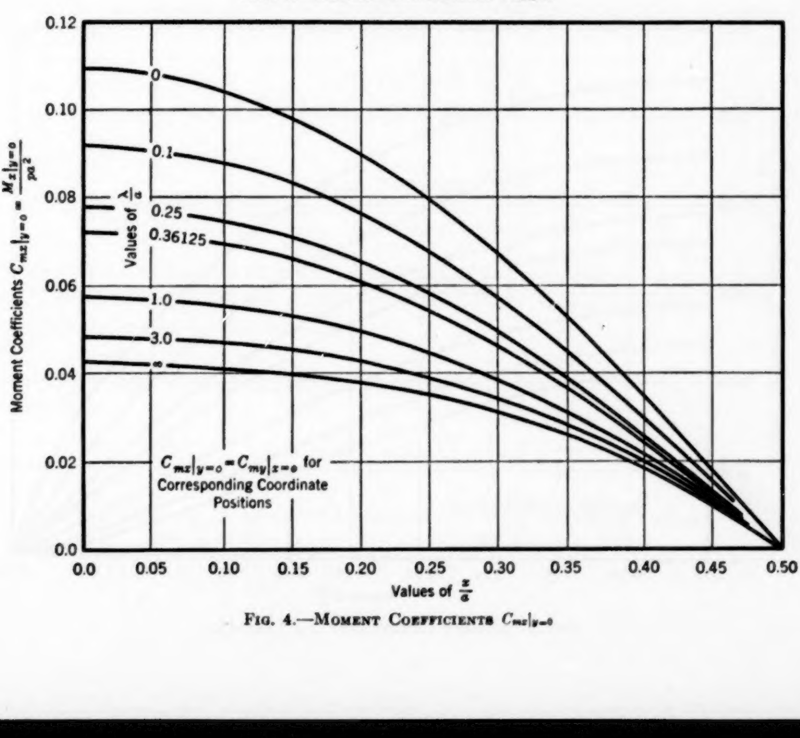

Fig. 4.—Moment Coefficients $C_{mx}|_{y=0}$

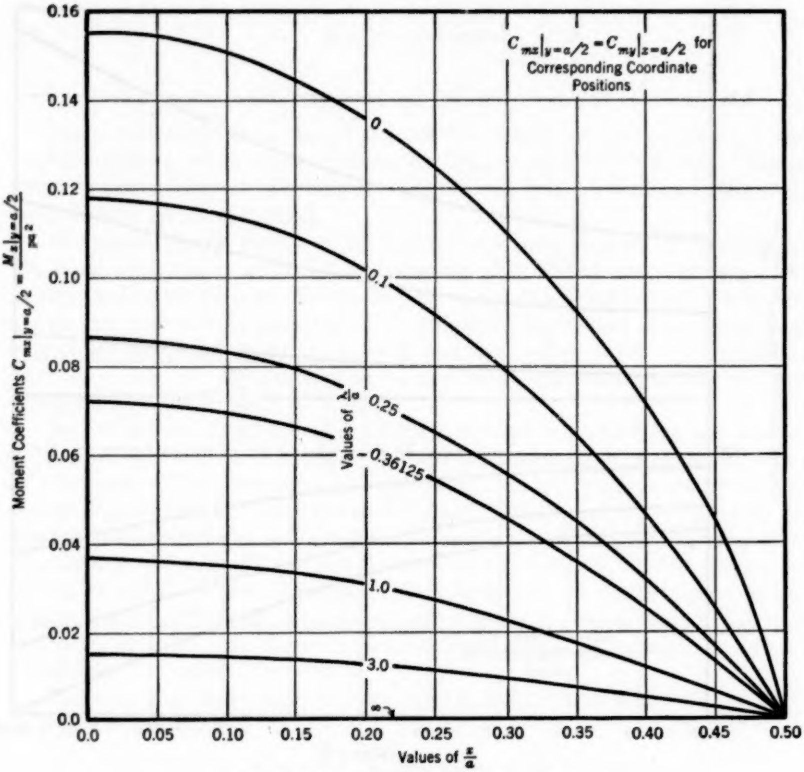

Fig. 5.—Moment Coefficients $C_{mx}|_{y=a/2}$

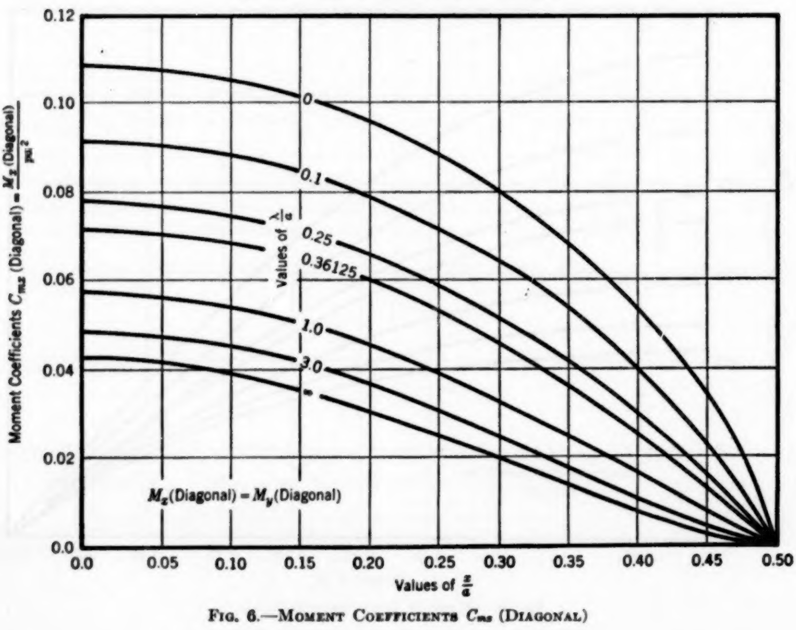

Fig. 6.—Moment Coefficients C_{mx} (Diagonal)

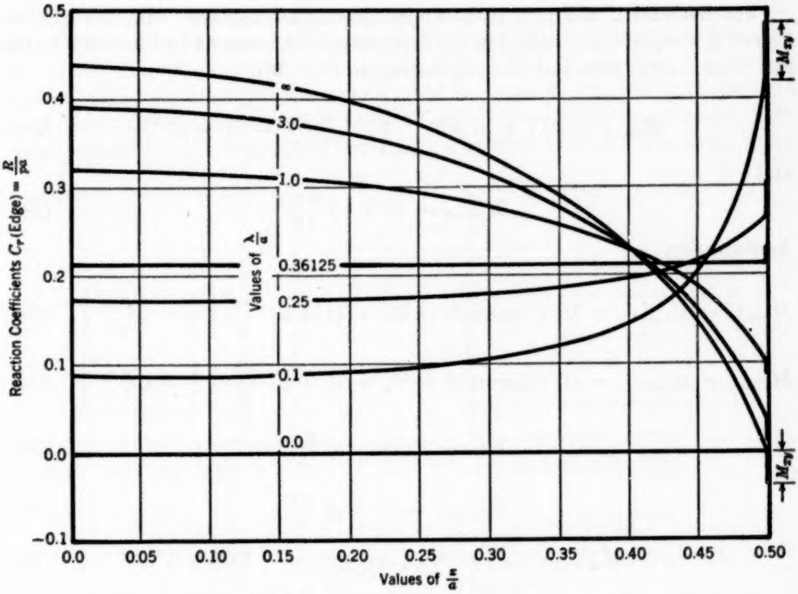

FIG. 7.—REACTION COEFFICIENTS C_r (EDGE)

sides of Eqs. 20 vanish, and hence in all these cases the trivial values $B_{2m} = B_{1m} = B_m = 0$ with $m = 1, 3, \cdots$, are sufficient to satisfy the equations for the boundary conditions. For all these cases, the corresponding constants obtained are

$$A_{2m} = \frac{s(m)\ 2\ \mu^p \operatorname{sech} \frac{m\ \pi\ b}{2\ a}}{(1 - \mu)\ m^5\ \pi^5} \qquad (26a)$$

and

$$A_{1m} = \frac{s(m)\ 2\ \mu^p \operatorname{sech} \frac{m\ \pi\ a}{2\ b}}{(1 - \mu)\ m^5\ \pi^5} \qquad (26b)$$

The terms containing A_m which appear in the various differential expressions for moment and reaction are further simplified for the values indicated by Eqs. 26, with the aid of the relationship

$$a^{-1} \sum_{m=1,3,\cdots}^{\infty} s(m)\ \alpha^{-3} \operatorname{sech} \alpha \frac{b}{2} \cosh \alpha\ y \cos \alpha\ x$$
$$= b^{-1} \sum_{m=1,3,\cdots}^{\infty} s(m)\ \beta^{-3} \operatorname{sech} \beta \frac{a}{2} \cosh \beta\ x \cos \beta\ y$$
$$+ a^{-1} \sum_{m=1,3,\cdots}^{\infty} s(m)\ \alpha^{-3} \cos \alpha\ x - b^{-1} \sum_{m=1,3,\cdots}^{\infty} s(m)\ \beta^{-3} \cos \beta\ y \qquad (27)$$

The derivation of Eq. 27 is shown in Appendix I, together with the application of this equation in reducing the expressions for moment and reaction to the following simple formulas (corresponding to Eqs. 25).

$$M_y|_{y=0} = 2\,(1+\mu)\,p\,b^2 \sum_{m=1,3,\ldots}^{\infty} \frac{s(m)}{m^3\,\pi^3} = (1+\mu)\,\frac{p\,b^2}{16} \qquad (28a)$$

and

$$M_x|_{x=0} = (1+\mu)\,\frac{p\,a^2}{16} \qquad (28b)$$

Analogously,

$$M_x|_{y=0} = M_x|_{y=b/2} = M_x\,(\text{diagonal}) = M_x = (1+\mu)\,\frac{p\,a^2}{4}\left[\frac{1}{4} - \left(\frac{x}{a}\right)^2\right] \qquad (29a)$$

$$M_y|_{x=0} = M_y|_{x=a/2} = M_y\,(\text{diagonal}) = M_y = (1+\mu)\,\frac{p\,b^2}{4}\left[\frac{1}{4} - \left(\frac{y}{b}\right)^2\right] \qquad (29b)$$

$$V_y|_{y=b/2} = (1-\mu)\,\frac{p\,b}{4} \qquad (30a)$$

$$V_x|_{x=a/2} = (1-\mu)\,\frac{p\,a}{4} \qquad (30b)$$

$$M_{xy}|_{x=a/2,\; y=b/2} = M_{xy}|_{y=b/2,\; x=a/2} = -\tfrac{1}{8}\,\mu\,p\,a\,b \qquad (31)$$

Eqs. 28 to 31 can be corroborated by the procedure previously described for sets of numerical evaluations, with numerical values determined from Eqs. 25 substituted for the constants A_m. A polynomial expression for w, conforming to Eqs. 25, and Eqs. 28 to 31, constructed independently of the general solution, is provided in Appendix I.

Experimental Verification

Experimental verification of the analysis previously presented was sought by two introductory tests of plates simply supported on beams of arbitrary flexural rigidity. Glass and gypsum plaster were selected as plate materials, because of the ideal linearity of the stress-strain relationship to the point of rupture for both materials. It was thought that the test results could be simplified by the use of plaster that would allow for the casting of the model with the supporting beams embedded symmetrically within the slab thickness. Also, from preliminary tests with sample specimens, it was concluded that commerical resistance type strain gages are convenient to use with plaster models and yield consistent results.

Model Specifications and Test Procedure.—The test model dimensions and the details of support are shown in Fig. 8. The uniform loading was applied in each test by water contained in a rubberized fabric tank, placed in contact with the plate. The tank portions extending beyond the panel boundaries were supported independently of the panel in order to eliminate any vertical component of pull on the panel edges. Strains in the plate and beam surfaces were observed by means of SR-4 resistance strain gages, type A-5, bonded to the plates in the positions shown in the plan views.

Test 1.—The plate material consisted of commercial plate glass, nominally ¼-in. thick. The four supporting bars were of steel, with mitre cut at the ends and arranged in a frame, as shown in Fig. 7(*b*). The supports for the bars at each corner of the frame consisted of a small disc, which was laid flat on a rigid plane base. For this preliminary test, the gages were applied on one side of the glass plate and of the supporting bar. The glass was first placed with gages on

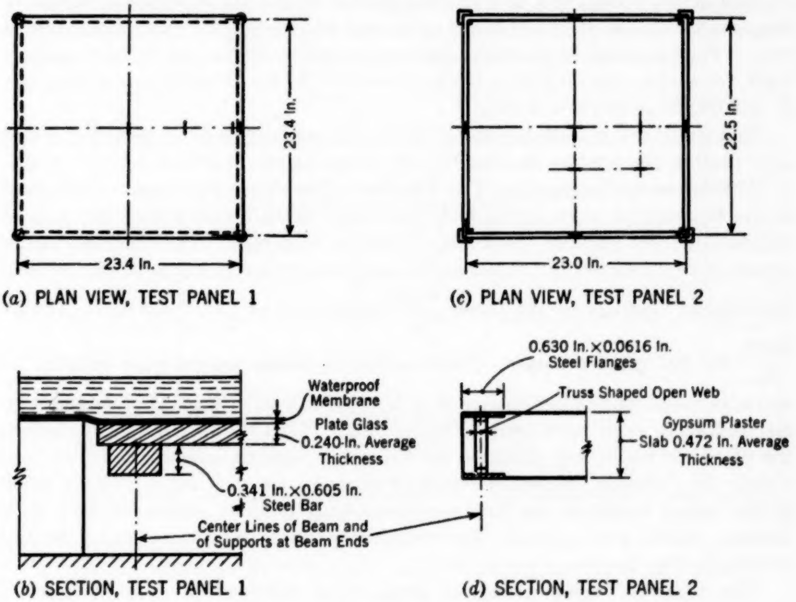

FIG. 8.—DETAILS OF TEST PANELS

the compression side. After taking initial readings, strain observations were recorded corresponding to three increments of water head. The water was then removed, the plate reversed, successive increments of load reapplied, and corresponding tensile strains recorded.

Test 2.—The gypsum plaster used for the plate material was of a commercial variety with a water-plaster ratio of 70:100 by weight. The supporting beams, which were fabricated with open webs, were embedded in the cast slab along the edges, and were connected at the panel corners to small blocks resting on jacks. Strain gages were applied on the top and bottom surfaces of the panel and supporting beams, usually in pairs at each gage station. Other gages, not indicated on the figure, were applied on the panel close to the beams, and on the beams close to the ends, to determine whether simple support was realized.

Successive increments of water head were applied and corresponding strain readings recorded. Following each increment of head, the water was entirely removed to ascertain whether the strain readings reverted to the initial zero values with each release of the load.

The modulus of elasticity of the plaster was determined by specimens in the shape of small flat beams cast from the same batch as used for the test panel, and also by a specimen cut out from the test panel at the conclusion of the experiment. From tests of these specimens in flexure, a mean value was determined for the modulus of elasticity of the plaster of 1.22×10^6 lb per sq in. Poisson's ratio was determined experimentally to be 0.14.

Test Model Values of λ/a.—For the glass of Test 1 the modulus of elasticity was assumed to be 10×10^6 lb per sq in. and Poisson's ratio was assumed to be 0.21. The modulus of elasticity for steel is 30×10^6 lb per sq in. Accordingly, for a glass plate 0.240 in. thick, $N = 0.00120\ E$. For the supporting bar, $I_1 = 0.00200$ and $\lambda/a = 0.214$.

In a similar way, with the aid of the experimentally derived values of E and μ for plaster, the relative flexural rigidity of the plaster model was $\lambda/a = 0.407$.

Evaluation of Comparative Test Results.—The strain readings, as observed in the laboratory, were reduced to equivalent mean strain values per unit of applied load, for uniform thickness of plate in each test. The observed strain values were corrected for variation in actual thickness h_1 by multiplication of the original readings by the factor $\frac{0.24}{h_1}$, in the case of $\frac{1}{4}$-in. glass plate, and by $\frac{0.472}{h_1}$ for the plaster plate. These corrected strain values were reduced to strain per unit pound per square inch by application of numerical factors corresponding to each increment of water head. Finally, for each gage station, the algebraic mean was obtained for the corresponding compression ($-$) and tensile ($+$) strains, corrected for thickness and for unit load. These mean strain values comprise the final experimentally derived strains at each gage station, which were plotted, for comparison with the theoretically derived strains, in Fig. 9.

The algebraic mean values of strain thus obtained in part eliminate a T-beam effect, whereby the strains on the top surface of the plate were consistently higher than the strains on the bottom surface, at the same gage stations. This effect was found during both tests and is attributed to the design of the panel supporting members. Accordingly, the test results shown in Fig. 9 include only algebraic mean values derived from gage readings where available in pairs—one reading for the top and one reading for the bottom surface, respectively, of plate or beam at each gage station.

For Test 2, the values plotted are the averages of these algebaric mean values for corresponding positions where available on two orthogonal axes, for various intensities of load. These average strain values further eliminate effects which are unsymmetrical with respect to the two axes.

Theoretically Derived Strain Values.—The value $\frac{\lambda}{a} = 0.214$, evaluated by means of the dimensions and elastic constants of model 1, agrees closely with $\frac{\lambda}{a} = 0.2125$, for which value of the parameter a numerical example was solved. Hence, corresponding to $\frac{\lambda}{a} = 0.2125$, theoretical strains $\epsilon_y|_{y=0}$, normal to an

axis of the plate, were computed for points generally along the axis. These computations can be made with the aid of moment coefficients computed for corresponding positions, by means of the relationship

$$\epsilon_y = \frac{h}{2}\frac{\partial^2 w}{\partial y^2} = \frac{h}{2\,N}\frac{a^2}{(1-\mu^2)}(C_{my} - \mu\, C_{mx}) \qquad (32)$$

in which substitution is made for the dimensions h, a, and the ratio μ to conform to the test model. Theoretically derived strains thus computed for the conditions of Test 1 are plotted in Fig. 9 for comparison with test results.

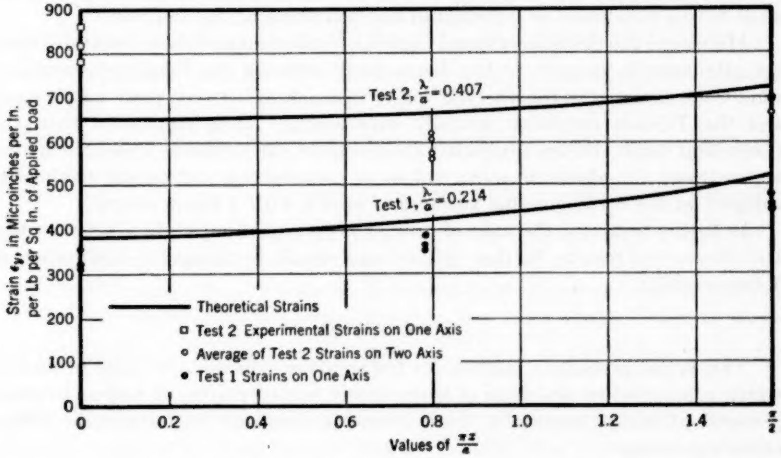

FIG. 9.—A COMPARISON OF TEST AND THEORETICAL STRAINS

Similarly, $\frac{\lambda}{a} = 0.407$, for model 2, is approximately equal to the parameter value 0.36125, for which the moment coefficients are expressed by exact formulas given by Eqs. 28 and 29. Hence, the moment coefficients for $\frac{\lambda}{a} = 0.407$ were obtained by use of Eqs. 28 and 29, and corrected by interpolation in the sets of coefficients plotted in Figs. 2 to 5. The strains $\epsilon_y|_{y=0}$, for the conditions of Test 2, were then computed by means of Eq. 32 and plotted in Fig. 9.

Comparison Between Test and Theoretical Values.—For Test 1, the average deviation of the test strains plotted in Fig. 9, with respect to the corresponding theoretical values represented by the curve, is evaluated as 12.6%, based on the theoretical results.

For Test 2, the average deviation between respective theoretical and test values is 9.1%, where these test values represent averages derived from gage readings at corresponding positions on two orthogonal axes of the model. However, the average deviation thus computed does not include a considerable deviation, amounting to 23.6% for a position on the test model where only uniaxial strains were available. At this position, the large deviation represents, in part, an apparent error due the unsymmetrical strains experienced in model 2.

The curves of Fig. 9 refer only to points along the axes parallel to the panel edges. For the station on the diagonal axis, shown on the panel diagram at $\frac{\pi x}{a} = 0.8$ (approximately the quarter point) the theoretical and experimental values are, respectively, 476 and 466 microinches per in. per lb per sq in. of applied load, where however, the experimental values are based only on uniaxial gage readings.

Test Conclusions.—The average deviations evaluated correspond to average test strain values which are 87% to 91% of the respective theoretical values. This numerical comparison, together with the graphical comparison of Fig. 9, leads to the conclusion of substantial corroboration of the analysis.

Moreover, the deviations found between theoretical and experimental strains are attributable, in part, to the discrepancy between the simplified boundary conditions assumed in the analysis (of vertical forces between plate and beams) and the T-beam condition actually experienced. It is concluded that the theoretical strain values computed according to the methods presented herein approximate the algebraic mean values of compressive and tensile strains developed on the corresponding surfaces of panels with T-beam action.

In future tests, for the sake of greater accuracy of correlation between test and theoretical results, further precautions should be taken for elimination of T-beam action.

Conclusions

This paper presents a method for the solution of boundary value problems, which is adapted to problems of transversely loaded plates, as well as to other physical problems, especially those involving harmonic or biharmohic differential equations.

The method is applied to the formulation of a solution for the problem of the rectangular plate, supported on beams along all the edges, for general values of parameters expressing the flexural and torsional rigidities of these beams. Most of the solutions in the literature on this subject apply to limit values of these parameters. As far as is known, the solution of this paper has not been obtained by other investigators.

The formal solution comprises an infinite series of terms with constant coefficients, dependent on the boundary conditions, as in most solutions of plate problems in literature. However, the determination of these constants involves the solution of sets of simultaneous linear equations. These constants, once obtained for the assigned values of the parameters, and for the assigned plate proportions, serve thereafter for computation of stress and deflection coefficients for various positions in the corresponding panels. It is found that the convergence of the constants A_m and B_m with $m = 1,3,\cdots$ is very rapid. These constants, as used in expressions for shear and moment, occur in combinations of terms, the convergence of which, in extreme cases, is slow. In these extreme cases, however, the accuracy is still estimated as adequate for design or graphical purposes. Similar conclusions as to accuracy apply to the limit case, at which $\lambda \rightarrow \infty$, the solution for which is identical with the classic solution for nonflexural supports.

The detailed application of the formal solution, complete with numerical results corresponds to the problem of simple, flexural support under uniform load. For this problem, a comparative numerical investigation was conducted of moment and reaction coefficients in relation to flexural rigidity of supporting beams. The graphs clearly show the considerable moments developed along the edges which comprise a characteristic feature of flexural support. Thus, in all square panels having a relative rigidity $\frac{\lambda}{a} < \frac{(1-\mu)^2}{2}$, the moments are maximum along the edges. The pattern of variation of reaction along the edges changes from concave downward to convex downward as the parameter lambda progresses from $\frac{\lambda}{a} > \frac{(1-\mu)^2}{2}$ to $\frac{\lambda}{a} < \frac{(1-\mu)^2}{2}$. As the parameter $\frac{\lambda}{a}$ decreases, equal increments in its value give rise to progressively larger changes in moment and shear.

For any rectangular panel, corresponding to the particular values $\frac{\lambda}{b} = \frac{(1-\mu)^2}{2}$ and $\frac{\lambda}{a} = \frac{(1-\mu)^2}{2}$, the derived expressions for moment reduce to simple exact formulas that represent parabolic cylinders superposed symmetrically with the axes of symmetry of the plate.

Maximum deflections, which are of general interest, are among the quantities that are easily obtained with extreme precision. From Table 2, these quantities are seen to vary from $0.026340 \frac{p\,a^4}{N}$ for $\lambda = 0$ to $0.004062 \frac{p\,a^4}{N}$ for $\lambda \rightarrow \infty$.

Introductory experimental results are found to be consistent with the theoretically computed corresponding data. The test results obtained also suggest improvements in the preparation and arrangement of panel specimens in future tests, so as to provide for increased accuracy of experimental corroboration of the theory.

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APPENDIX I. DERIVATION OF EQUATIONS

Since $w_1 = w_2$, application of the same differential operator to w_1 and w_2 yields

$$(w_1)_{xx} + (w_1)_{yy} = (w_2)_{xx} + (w_2)_{yy} \qquad (33)$$

Substitution in Eq. 33, by termwise differentiation of Eq. 24, results in Eq. 27.

The following equations are obtained from Eq. 27 by termwise differentiation or integration; that is, by the differentiation of Eq. 27 with respect to y,

$$\sum_{m=1,3,\cdots}^{\infty} \frac{s(m)}{b\,\beta^2} \sin\beta\,y - \sum_{m=1,3,\cdots}^{\infty} \frac{s(m)}{b\,\beta^2} \operatorname{sech}\beta\frac{a}{2}\cosh\beta\sin\beta\,y$$
$$= \sum_{m=1,3,\cdots}^{\infty} \frac{s(m)}{a\,\alpha^2} \operatorname{sech}\alpha\frac{b}{2}\sinh\alpha\,y\cos\alpha\,x \quad \text{(34a)}$$

and by the integration of Eq. 27 with respect to y,

$$-\sum_{m=1,3,\cdots}^{\infty} \frac{s(m)}{b\,\beta^4} \sin\beta\,y + \sum_{m=1,3,\cdots}^{\infty} \frac{s(m)}{a\,\alpha^3}\,y\cos\alpha\,x$$
$$+ \sum_{m=1,3,\cdots}^{\infty} \frac{s(m)}{b\,\beta^4} \operatorname{sech}\beta\frac{a}{2}\cosh\beta\,x\sin\beta\,y$$
$$= \sum_{m=1,3,\cdots}^{\infty} \frac{s(m)}{a\,\alpha^4} \operatorname{sech}\frac{\alpha\,b}{2}\sinh\alpha\,y\cos\alpha\,x \quad \text{(34b)}$$

and by the differentiation of Eq. 34b with respect to x,

$$-\sum_{m=1,3,\cdots}^{\infty} \frac{s(m)}{a\,\alpha^2}\,y\sin\alpha\,x + \sum_{m=1,3,\cdots}^{\infty} \frac{s(m)}{b\,\beta^3} \operatorname{sech}\beta\frac{a}{2}\sinh\beta\,x\sin\beta\,y$$
$$= -\sum_{m=1,3,\cdots}^{\infty} \frac{s(m)}{a\,\alpha^3} \operatorname{sech}\alpha\frac{b}{2}\sinh\alpha\,y\sin\alpha\,x \quad \text{(34c)}$$

It may be noted that Eqs. 23 for the constants B_{1m} and B_{2m} are the equivalent, except for a constant factor, of Eqs. 26 for the constants A_{1m} and A_{2m}. Hence, on substitution into the differential expressions for moment,

$$M_y = -N\,(w_{yy} + \mu\,w_{xx}) \quad \text{(35a)}$$

$$M_x = -N\,(w_{xx} + \mu\,w_{yy}) \quad \text{(35b)}$$

of Eq. 4, Eqs. 26, and the trivial values $B_{1m} = B_{2m} = 0$ in which $m = 1,3,\cdots$, it is found, with the aid of Eq. 27, that all the terms containing A_{1m} and A_{2m} vanish from the resulting expressions, which readily reduce to Eqs. 28 and 29.

Similarly, Eq. 31, for twisting moment, is readily obtained with the aid of Eq. 34c, from the differential expression $M_{xy} = (1-\mu)\,N\,w_{xy}$.

Also, Eqs. 30 are readily obtained with the aid of Eq. 34a from the differential expressions represented by the left-hand sides of Eqs. 5 and 6.

The polynomial expression for w is constructed to satisfy the boundary condition (Eq. 10) for simple support:

$$w = \frac{p}{32\,(1-\mu)\,N}\Big[(a^2 - \mu\,b^2)\,x^2 - \tfrac{2}{3}\,x^4$$
$$+ (b^2 - \mu\,a^2)\,y^2 - \tfrac{2}{3}\,y^4 + 4\,\mu\,x^2\,y^2 - k_0\Big] \quad \text{(35c)}$$

in which $k_0 = \frac{5}{24}\,(a^4 + b^4) - \mu\,\frac{a^2\,b^2}{4}$.

It is found that Eqs. 5 and 6 for equilibrium are also satisfied by Eq. 35c, provided that the values for λ_2 and λ_1 conform to Eqs. 25. Eqs. 28 to 31 follow from Eq. 35c.

APPENDIX II. NOTATION

The following letter symbols, adopted for use in the paper and for the guidance of discussers, conform essentially with "American Standard Letter Symbols for Structural Analysis" (ASA Z10.8—1949), prepared by a committee of American Standards Association with ASCE participation, and approved by the Association in 1949.

$A_{1m}, A_{2m}, B_{1m}, B_{2m}$ = arbitrary constant coefficients corresponding to each value of the index m of the series comprising the primitive of the plate differential equation.

a = panel span length measured parallel to the x-axis.

b = panel span length measured parallel to the y-axis.

C_{mx} = moment coefficient for the moment M_x.

C_{my} = moment coefficient for the moment M_y.

C_r = reaction coefficient.

c = column matrix comprising the unknowns B_m with $m = 1,3,\cdots$.

E = modulus of elasticity of the plate material.

E_1 = modulus of elasticity of supporting beam material at edges $x = \pm \frac{a}{2}$.

E_2 = modulus of elasticity of supporting beam material at edges $y = \pm \frac{b}{2}$.

H_1 = torsional rigidity of the supporting beams at edge $x = \pm \frac{a}{2}$.

H_2 = torsional rigidity of the supporting beams at edge $y = \pm \frac{b}{2}$.

h = depth or thickness of the plate material.

I = unit diagonal in a matrix.

I_1 = moment of inertia of the cross-sectional areas of the supporting beams at edge $x = \pm \frac{a}{2}$.

I_2 = moment of inertia of the cross-sectional areas of the supporting beams at edge $y = \pm \frac{b}{2}$.

k_0, k_1, k_2 = constants used in the general equation.

l = matrix of the numerical coefficients on the left-hand side of Eq. 21b.

M_x = bending moment per unit length about axis parallel to the y-axis, considered to be positive when causing compression in the top fibers of the plate or beams.

M_y = bending moment per unit length about axis parallel to the x-axis, same sign convention as for M_x.

M_{xy} = twisting moment per unit length of section of plate normal to the x-axis.

M_{yx} = twisting moment per unit length of section of plate normal to the y-axis.
m = index of summation.
N = flexural rigidity of the plate material.
n = index of summation.
p = intensity of the distributed load, positive downward.
r = right-hand side of Eq. 21*b*.
$s(m) = (-1)^{(m-1)/2}$ with $m = 1,3,\cdots$.
$s(n) = (-1)^{(n-1)/2}$ with $n = 1,3,\cdots$.
t = matrix l minus the unit diagonal I.
w = deflection of the plate, positive downward.
x, y, z = Cartesian coordinates.
$\alpha = \frac{m\pi}{a}$.
$\alpha' = \frac{n\pi}{a}$.
$\beta = \frac{m\pi}{b}$.
$\beta' = \frac{n\pi}{b}$.
ϵ = strain in the plate.
$\lambda_1 = \frac{E_1 I_1}{N}$ at edge $x = \pm \frac{a}{2}$.
$\lambda_2 = \frac{E_2 I_2}{N}$ at edge $y = \pm \frac{b}{2}$.
K_1 = ratio of the torsional rigidity H_1 to the flexural rigidity N at edge $x = \pm \frac{a}{2}$.
K_2 = ratio of the torsional rigidity H_2 to the flexural rigidity N at edge $y = \pm \frac{b}{2}$.
μ = Poission's ratio.
ϕ = a particular integral.

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